TOPOLOGICAL MULTIGROUPS

By

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Abstract of Dissertation Presented to the Graduate Council of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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The study of discrete set-valued multiplications on a set is originated and developed by O. Ore, M. Dresher, and J. E. Eaton. On the other hand, the topological observations of set-valued functions have been investigated extensively over the past forty years. Enough information about the relationship between a space X and 2^X, the space of all nonvoid closed subsets of X, was given by E. Michael. In the case where X is a continuum, J. L. Kelley investigated numerous properties of 2^X. W. L. Strother studied the continuity of set-valued functions extensively in 1951. Furthermore, A. D. Wallace, S. Eilenberg, and L. E. Ward contributed heavily to the theory of fixed point property for set-valued functions.

Recently, A. D. Wallace suggested an idea combining the above two algebraic and topological concepts together. No literature concerning topological algebra based on a set-valued multiplication could readily be found. This dissertation is devoted to the investigation of basic theory of binary set-valued topological algebra and, in particular, the properties of set-valued multiplications on an interval.

A multi-mob is a nonvoid Hausdorff space together with a continuous associative set-valued multiplication.

In chapter I, definitions and examples of multi-mobs are given. Also a condition for the existence of a scalar idempotent is found.

In chapter II, the elementary properties of submobs, ideals, and homomorphisms are investigated.

In chapter III, it is shown that the multi-semilattice on an interval in which an end point is a zero has the exact structure of a topological semilattice. More on multi-semilattices and standard multi-mobs are studied.

In chapter IV, some properties of multi-actions are investigated by using their induced semigroup actions.

CHAPTER I

MULTI-MOB

1. Vietoris Topology

Throughout this paper, all spaces in consideration are assumed to be Hausdorff. Let X be a space and let 2^X be the set of all nonvoid closed subsets of X. With each subset A of X we associate the following subsets of 2^X:

$$L(A) = \{ B \in 2^X \mid B \subset A \}, \quad M(A) = \{ B \in 2^X \mid B \cap A \neq \Box \}.$$

The Vietoris topology for 2^X is the topology having the family

$$\{ L(U) \mid U = U^o \subset X \} \cup \{ M(V) \mid V = V^o \subset X \}$$

as a subbase of it. The Vietoris topology for 2^X is also known by the names "finite topology" and "exponential topology" [12]. This is equivalent to the topology used by Frink [8] and Strother [18].

If $\{ A_j \mid j = 1,\ldots,n \}$ is a finite collection of

subsets of a space X, then $< A_1, \ldots, A_n >$ is defined to be the set of all nonvoid closed subsets of X which are covered by $\{ A_j \mid j = 1, \ldots, n \}$ and intersect every member of $\{ A_j \mid j = 1, \ldots, n \}$, i.e.,

$$< A_1, \ldots, A_n > = \{ B \in 2^X \mid B \subset \bigcup_{j=1}^{n} A_j \text{ and } B \cap A_j \neq \emptyset \text{ for each } j \}.$$

Let $U_1, \ldots, U_n; V_1, \ldots, V_m$ be subsets of a space X and let $U = U_1 \cup \ldots \cup U_n$, $V = V_1 \cup \ldots \cup V_m$. Since $2^X = < X >$ and $< U_1, \ldots, U_n > \cap < V_1, \ldots, V_m > = < U_1 \cap V, \ldots, U_n \cap V, U \cap V_1, \ldots, U \cap V_m >$, the collection of the form $< W_1, \ldots, W_p >$ with $W_1, \ldots, W_p$ being open in X forms a base for the Vietoris topology on 2^X.

Since $L(A) = 2^X - M(X - A)$ and $M(A) = 2^X - L(X - A)$, $L(A)$ and $M(A)$ are open (closed) if A is open (closed).

The following lemma concerning the relation between the topology of X and the Vietoris topology of 2^X may be found in [12]. We shall give the proof of (2), in the next lemma, for the interest of its techniques.

<u>Lemma 1.1</u>. Let X be a space. Suppose 2^X has the Vietoris topology and $C(X)$, the set of all nonvoid compact subsets of X, carries the relative topology of 2^X. Then

(1) X is regular if and only if 2^X is Hausdorff,

(2) X is compact if and only if 2^X is compact,

(3) X is metrizable if and only if $C(X)$ is metrizable.

Proof of (2). Suppose 2^X is compact. Let $\{L(U_\lambda) \mid \lambda \in \Lambda\}$ be an open cover for X. Then $\{<X, U_\lambda> \mid \lambda \in \Lambda\}$ is an open cover for 2^X, and there is a finite subcover $\{<X, U_1>, \ldots, <X, U_n>\}$ for 2^X. But then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{L(U_\lambda) \mid \lambda \in \Lambda\}$ for X.

Now suppose X is compact. Let $\{L(U_\lambda) \mid \lambda \in \Lambda\} \cup \{M(V_\gamma) \mid \gamma \in \Gamma\}$ be an open cover for 2^X by the subbase members of the topology for 2^X. Let $V = \bigcup\{V_\gamma \mid \gamma \in \Gamma\}$. Then $X - V \subset U_{\lambda o}$ for some $\lambda_o \in \Lambda$ since $X - V \in M(V_\gamma)$ for any $\gamma \in \Lambda$. Therefore $\{V_\gamma \mid \gamma \in \Gamma\}$ is an open cover for the compact set $X - U_{\lambda o}$ and there are $V_1, \ldots, V_n \in \{V_\gamma \mid \gamma \in \Gamma\}$ such that $X - U_{\lambda o} \subset V_1 \cup \ldots \cup V_n$. Now it is clear that the family $\{L(U_{\lambda o}), M(V_1), \ldots, M(V_n)\}$ covers 2^X, and 2^X is compact by Alexander's subbase theorem.

If (X, d) is a metric space, then the Hausdorff metric [26] ρ for $C(X)$ is given by

$$\rho(A, B) = \max(\sup\{d(a, B) \mid a \in A\}, \sup\{d(A, b) \mid b \in B\}).$$

Theorem 1.2. If X is a metric space, then the Hausdorff metric topology for $C(X)$ is equivalent to the relative topology on $C(X)$ induced by the Vietoris topology of 2^X.

Proof. Let $\varepsilon > 0$ be given. Let $A \in C(X)$ and let $\mathfrak{N}_\varepsilon(A)$ denote the ε-neighborhood of A in $(C(X), \rho)$, i.e., $\mathfrak{N}_\varepsilon(A) = \{B \in C(X) \mid \rho(A, B) < \varepsilon\}$. Since A is compact in X, there exist $a_1, \ldots, a_n \in A$ and $N_{\varepsilon/2}(a_1), \ldots, N_{\varepsilon/2}(a_n)$, the respective $\varepsilon/2$-neighborhoods of $a_1, \ldots, a_n$ in X, such that $A \subset \bigcup_{j=1}^n N_{\varepsilon/2}(a_j)$. Then $A \in \langle N_{\varepsilon/2}(a_1), \ldots, N_{\varepsilon/2}(a_n)\rangle$. If $B \in \langle N_{\varepsilon/2}(a_1), \ldots, N_{\varepsilon/2}(a_n)\rangle$ then $B \subset \bigcup_{j=1}^n N_{\varepsilon/2}(a_j)$ and $B \cap N_{\varepsilon/2}(a_j) \neq \square$ for each j. If $b \in B$, $d(a_i, b) < \varepsilon/2$ for some i. Therefore $d(A, b) < \varepsilon/2$. If $a \in A$, then $d(a, a_j) < \varepsilon/2$ for some j. Since $B \cap N_{\varepsilon/2}(a_j) \neq \square$, $d(a_j, b') < \varepsilon/2$ for some $b' \in B$. But then $d(a, b') \leq d(a, a_j) + d(a_j, b') < \varepsilon$, i.e., $d(a, B) < \varepsilon$. By the definition of Hausdorff metric, $\rho(A, B) < \varepsilon$. Therefore $B \in \mathfrak{N}_\varepsilon(A)$.

Conversely, let $U_1, \ldots, U_n$ be open in X and let $A \in \langle U_1, \ldots, U_n\rangle$. If $U = \bigcup_{j=1}^n U_j$, then $A \subset U$ and $A \cap U_j \neq \square$ for each j. Let $2\varepsilon = d(A, X - U) > 0$. Since A is compact, there exist $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{j=1}^n N_\varepsilon(a_j) \subset \bigcup_{j=1}^n N_{2\varepsilon}(a_j) \subset U$. Let $B \in \mathfrak{N}_{\varepsilon/2}(A)$ and $b \in B$. Then $d(A, b) < \varepsilon/2$. Since A is compact, $d(a, b) < \varepsilon/2$ for some

$a \in A$. Since $A \subset \bigcup_{j=1}^{n} N_\varepsilon(a_j)$, $d(a, a_i) < \varepsilon$ for some i and $d(b, a_i) < 2\varepsilon$. Then $b \in U$ and $B \subset U$. Now let $a \in U_i \cap A$ and choose $\varepsilon' > 0$ such that $2\varepsilon' < \varepsilon$ and $N_{\varepsilon'}(a) \subset U_i$. If $B \in \mathfrak{N}_{\varepsilon'}(A)$, then $d(a, B) < \varepsilon$ and hence there exists $b \in B$ such that $d(a, b) < \varepsilon'$. Then $b \in U_i$, and $B \cap U_j \neq \square$ for each j, i.e., $\mathfrak{N}_{\varepsilon'}(A) \subset \langle U_1, \ldots, U_n \rangle$.

2. <u>Multi-Valued Functions</u>

Henceforth, 2^X is always assumed to have the Vietoris topology. The following lemma is a criterion for the continuity of multi-valued functions and is found in [12].

<u>Lemma 1.3.</u> A function $f\colon X \longrightarrow 2^Y$ is continuous if and only if the following conditions are satisfied:

(1) $\{x \in X \mid f(x) \cap A \neq \square\}$ is open in X whenever A is open in Y,

(2) $\{x \in X \mid f(x) \cap A \neq \square\}$ is closed in X whenever A is closed in Y.

<u>Proof</u>. If f is continuous, if V is an open subset of Y, then $\{x \in X \mid f(x) \cap V \neq \square\} = f^{-1}(M(V))$ is open in X. If A is a closed subset of Y, then $\{x \in X \mid f(x) \cap A \neq \square\} = X - f^{-1}(L(Y - A))$ is closed in X.

Conversely, if V is an open subset of Y, then $f^{-1}(M(V)) = \{x \in X \mid f(x) \cap V \neq \square\}$ is open by using (1), above. By using (2), above, $\{x \in X \mid f(x) \cap (Y - V) \neq \square\}$ is closed. Then $\{x \in X \mid f(x) \cap (Y - V) \neq \square\} = f^{-1}(M(Y - V))$ $= f^{-1}(2^Y - L(V)) = X - f^{-1}(L(V))$ is closed in X and $f^{-1}(L(V))$ is open in X. Therefore f is continuous.

Corollary 1.4. Let $f: X \longrightarrow Y$ be a function. Define $\bar{f}: X \longrightarrow 2^Y$ via $\bar{f}(X) = \{f(x)\}$. Then $\bar{f}$ is continuous if and only if f is continuous.

Proof. Let $A \subset Y$. Then $\{x \in X \mid \bar{f}(x) \cap A \neq \square\} = f^{-1}(A)$.

Corollary 1.5. A function $f: X \longrightarrow 2^Y$ is continuous if and only if the following two conditions hold:

(1) V being open in Y and $f(x_o) \cap V \neq \square$ imply that there is an open set U in X containing x_o with $f(x) \cap V \neq \square$ for each $x \in U$,

(2) V being open in Y and $f(x_o) \subset V$ imply that there is an open set U in X containing x_o with $f(x) \subset V$ for each $x \in U$.

Remark. The Vietoris topology for 2^X can be extended to $\mathcal{P}(X)$, the power set of X (see [12]). However, if Y is a T_1 space and if $f: X \longrightarrow \mathcal{P}(Y)$ is continuous then $f(X) \subset 2^Y$ [18]. Since all spaces are assumed to be Hausdorff, we lose nothing by restricting observations to 2^X rather than to $\mathcal{P}(X)$.

3. Multi-Mob

Definition 1.6. A multi-mob is a nonvoid Hausdorff space S together with a continuous function $S \times S \longrightarrow 2^S$ (whose value at (x, y) will be denoted by xy) satisfying $(xy)z = x(yz)$ for all $x, y, z \in S$. AB is defined to be the union $\bigcup \{ ab \mid a \in A,\ b \in B \}$ for $A, B \subset S$.

Examples 1.7.

(1). Any space X is a multi-mob under the multiplication

$$xy = \{x, y\} \text{ for each } x, y \in X.$$

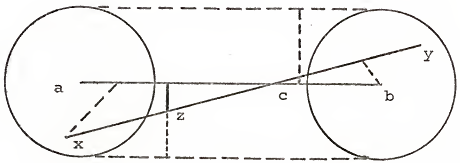
Proof. Let $A \subset X$ and let $B = \{ (x, y) \in X \times X \mid xy \cap A \neq \square \}$. It is readily seen that $B = A \times X \cup X \times A$, and B is open (closed) whenever A is open (closed). Therefore the multiplication is continuous by using Lemma 1.3. Also the multiplication is associative since $(xy)z = \{x, y, z\} = x(yz)$ for all $x, y, z \in X$.

(2). Any topological semigroup ([9],[13],[19]) is a multi-mob under the semigroup multiplication by using Corollary 1.4.

(3). Let X be a plane with usual topology. By defining xy to be the segment joining x and y for each pair of distinct points $x, y \in X$, and x^2 to be the

singleton $\{x\}$, X becomes a multi-mob under this multiplication.

<u>Proof</u>. By Theorem 1.2, the Hausdorff metric topology may be used for $C(X)$ since $xy \in C(X)$ for each $x, y \in X$. Let $(a, b) \in X \times X$ and let $\mathfrak{N}_\varepsilon(ab) = \{C \in C(X) \mid \rho(ab, C) < \varepsilon\}$. Let $N_{\varepsilon/2}(a)$ and $N_{\varepsilon/2}(b)$ be the $\varepsilon/2$-neighborhoods of a and b in X respectively. Let $x \in N_{\varepsilon/2}(a)$ and $y \in N_{\varepsilon/2}(b)$. Since $d(ab, z) \leq \varepsilon/2$ and $d(c, xy) \leq \varepsilon/2$ for each $z \in xy$ and each $c \in ab$, $\sup\{d(c, xy) \mid c \in ab\} < \varepsilon$ and $\sup\{d(ab, z) \mid z \in xy\} < \varepsilon$. Therefore $\rho(ab, xy) < \varepsilon$ and $N_{\varepsilon/2}(a)N_{\varepsilon/2}(b) = \bigcup\{xy \mid x \in N_{\varepsilon/2}(a), y \in N_{\varepsilon/2}(b)\} \subset \mathfrak{N}_\varepsilon(ab)$ so that the multiplication is continuous. The associativity is evident, and X is a multi-mob under the given multiplication.



(4). Let $X = [a, b]$ be the real closed interval from a to b. Then X is a multi-mob under the multiplication

$$xy = [a, \min\{x, y\}].$$

Proof. Let $f: X \longrightarrow 2^X$ be the function defined by $f(x) = [a, x]$ for each $x \in X$. Let $c \in X$ and let $\mathfrak{N}_\varepsilon(f(c))$ be the ε-neighborhood of $f(c)$ in 2^X, i.e., $\mathfrak{N}_\varepsilon(f(c)) = \{E \in 2^X \mid \rho([a, c], E) < \varepsilon\}$. Let $N_{\varepsilon/2}(c)$ be the $\varepsilon/2$-neighborhood of c in X. If $x \in N_{\varepsilon/2}(c)$, then $d(y, [a, c]) \leq \varepsilon/2$ and $d(b, [a, x]) \leq \varepsilon/2$ for each $b \in [a, c]$ and for each $y \in [a, x]$, and therefore $\rho([a, c], [a, x]) < \varepsilon$ for each $x \in N_{\varepsilon/2}(c)$, i.e., $f(N_{\varepsilon/2}(c)) \subset \mathfrak{N}_\varepsilon(f(c))$. Hence f is continuous. Since $g: X \times X \longrightarrow X$, via $g(x, y) = \min\{x, y\}$, is continuous, fg is continuous and $fg(x, y) = [a, \min\{x, y\}]$. It is clear that $(xy)z = [a, \min\{x, y, z\}] = x(yz)$, and X is a multi-mob under the given multiplication.

(5). Let $X = [0, 1]$ be the real unit interval. Then X is a multi-mob under the multiplication

$$xy = [0, \text{the usual product of } x \text{ and } y].$$

(6). Let $X = [a, b]$ be the real closed interval from a to b. Then the following are also multi-mob multiplications:

$$xy = [a, x] \quad xy = [a, y], \quad xy = [x, b], \quad xy = [x, y].$$

(7). Let X be the unit disc. Define xy to be the segment joining the center of X to the point corresponding to the usual product of the complex numbers x and y in X. Then X becomes a multi-mob under this multiplication.

Definition 1.8. Let S be a multi-mob.

An element u of S is called a left unit if and only if $x \in ux$ for each x in S.

An element e of S is called an idempotent if and only if $e^2 = e$.

An element f of S is called a multi-idempotent if and only if $f \in f^2$.

An element s of S is called a left scalar if and only if sx is a singleton for each x in S.

An element u of S is called a left scalar unit if and only if u is a left scalar and a left unit, i.e.,

$$ux = x \text{ for each element } x \text{ in } X.$$

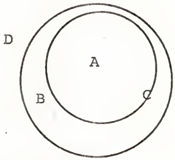
An element e of S is called a left scalar idempotent if and only if e is a left scalar and an idempotent.

In each definition, above, right and two-sided elements are defined analogously.

Remarks. It will be observed that no differentiation is made between x and $\{x\}$ if it is not convenient to do so and will not readily lead to confusion. It may be seen that if a multi-mob S contains a scalar unit then there is no other left or right unit in S [5][7]. The following example shows that a compact connected multi-mob does not necessarily have an idempotent. However, the existence of a scalar idempotent of a compact multi-mob will be shown under certain conditions.

Example 1.9. Let D be a disc. Choose another disc A in D and name the annulus B and the bounding circle C respectively as in the following figure. Define a multiplication on D as follows:

$$xy = \begin{cases} C \cup \{x, y\} & \text{if } x, y \in B \\ C \cup \{y\} & \text{if } x \in A,\ y \in B \\ C \cup \{x\} & \text{if } x \in B,\ y \in A \\ C & \text{if } x, y \in A. \end{cases}$$



Then D becomes a multi-mob under this multiplication and has no idempotent.

Theorem 1.10. If a compact multi-mob S contains a left (right, two-sided) scalar element, then S has a left (right, two-sided) scalar idempotent.

Proof. Let $\{S\}$ denote the subset $\{\{s\} \mid s \in S\}$ of 2^S. By Corollary 1.4, $\{S\}$ is closed in 2^S. Let A be the set of all left scalar elements of S. Then S is nonvoid by the hypothesis. For each x in S, let

$$A_x = \{ s \in S \mid sx \text{ is a singleton} \}.$$

Define a function $f\colon S \longrightarrow 2^S$ via $f(s) = sx$. Then f is continuous and hence $A_x = f^{-1}(\{S\})$ is closed in S. Since $A = \bigcap \{ A_x \mid x \in S \}$, A is a closed subset of S. It is clear that $s, t \in A$ implies $st \in A$. Thus A is a compact semigroup and therefore S has a left scalar idempotent.

CHAPTER II

IDEALS AND HOMOMORPHISMS

1. Submobs

Conventions. Throughout, S will denote a multi-mob and E will denote the set of all multi-idempotents of S. For subsets A, B of S, $A \circ B$ will denote the family $\{ab \mid a \in A, b \in B\}$, i.e., $AB \subset S$ and $A \circ B \subset 2^S$.

Theorem 2.1. Let A and B be compact subsets of S. If AB is contained in an open subset W of S, then there exist open subsets U and V of S such that

$$A \subset U, \quad B \subset V, \quad \text{and} \quad UV \subset W.$$

Proof. Since $AB \subset W$, $ab \subset W$ for each $a \in A$ and each $b \in B$, therefore, $A \circ B \subset L(W)$. Since $L(W)$ is open in 2^S and since the multiplication is continuous, by using Wallace's theorem ([10],[14],[20],[21]), there exist open subsets U and V of S such that

$$A \subset U, \quad B \subset V, \quad \text{and} \quad U \circ V \subset L(W).$$

By taking their unions, we have $UV \subset L(W) \subset W$.

Corollary 2.2. Let A be a compact subset of S and let $x \in S$. If Ax is contained in an open subset V of S, then there exists an open subset U of S such that $x \in U$ and $AU \subset V$.

Notations. For subsets A and B of S, it is convenient to write

$$A^{[-1]}B = \{x \in S \mid Ax \subset B\}, \quad A^{(-1)}B = \{x \in S \mid Ax \cap B \neq \square\},$$

$$BA^{[-1]} = \{x \in S \mid xA \subset B\}, \quad BA^{(-1)} = \{x \in S \mid xA \cap B \neq \square\}.$$

In the case where S is a topological semigroup, various forms of the proof of the following theorem have been given in [15] and [20].

Theorem 2.3. Let A and B be subsets of S. Then

(1) If A is compact and if B is open, then $A^{[-1]}B$ is open.

(2) If A is compact and if B is closed, then $A^{(-1)}B$ is closed.

(3) If A is compact, then $\{x \in S \mid B \subset Ax\}$ is closed.

Proof. (1). If $x \in A^{[-1]}B$, then $Ax \subset B$. Since

A is compact and since B is open, by Corollary 2.2, there exists an open subset V of S such that $x \in V$ and $AV \subset B$, i.e., $V \subset A^{[-1]}B$. Therefore $A^{[-1]}B$ is open in S. (2) and (3) may be proved by observing that $A^{[-1]}(S - B) = S - A^{(-1)}B$ and $\{x \in S \mid B \subset Ax\} = \bigcap\{A^{(-1)}b \mid b \in B\}$ respectively.

Definition 2.4. A nonvoid subset A of S is called a submob of S if and only if $A^2 \subset A$.

Lemma 2.5. Let X and Y be spaces and let f be a function from X into Y.

(1) f is continuous if and only if $f(A^*) \subset f(A)^*$ for all $A \subset X$.

(2) If f is continuous and if A is a subset of X whose closure is compact, then $f(A^*) = f(A)^*$ [10].

Lemma 2.6. Let X be a regular space and let $\mathcal{B}$ be a compact subset of 2^X. Then $\bigcup \mathcal{B}$ is closed in X [12].

Theorem 2.7. Let A and B be subsets of S. Then

(1) $A^*B^* \subset (AB)^*$.

(2) If S is regular and if A^* and B^* are compact, then $A^*B^* = (AB)^*$.

<u>Proof</u>. For $\mathfrak{B} \subset 2^S$, $\mathfrak{B}^-$ will denote the closure of $\mathfrak{B}$ in 2^S.

(1). Since $A^* \times B^* = (A \times B)^*$, by using the continuity of the multiplication and Lemma 2.5, $A^* \circ B^* \subset (A \circ B)^-$. Therefore, $A^*B^* \subset \bigcup (A \circ B)^-$. It will be shown that $\bigcup (A \circ B)^- \subset (AB)^*$. Suppose $x \notin (AB)^*$. Then there is an open subset U of S such that $x \in U$ and $U \cap AB = \square$, and $ab \subset S - U$ for each $a \in A$ and each $b \in B$, i.e., $A \circ B \subset L(S - U)$. It follows that $(A \circ B)^- \subset L(S - U)$ since $L(S - U)$ is closed in 2^S. Therefore $C \cap U = \square$ for each $C \in (A \circ B)^-$. Then $x \notin C$ for each $C \in (A \circ B)^-$, and hence $x \notin \bigcup (A \circ B)^-$. By contraposition, $A^*B^* \subset \bigcup (A \circ B)^- \subset (AB)^*$.

(2). By the compactness of A^* and B^*, $A^* \times B^* = (A \times B)^*$ is compact in $S \times S$. Again by using Lemma 2.5, $A^*B^* = \bigcup (A^* \circ B^*) = \bigcup (A \circ B)^-$. Since $(A \circ B)^-$ is compact in 2^S, by using Lemma 2.6, A^*B^* is closed in S. Therefore $(AB)^* \subset (A^*B^*)^* = A^*B^*$. By using (1), above, $A^*B^* = (AB)^*$.

<u>Theorem 2.8</u>.

(1) The closure of a submob of S is a submob of S.

(2) The intersection of a family of submobs of S is a submob of S if it is nonvoid.

(3) If S is regular, then the closure of a commutative submob of S is also a commutative submob of S.

Proof. (1). If $A^2 \subset A$, then $(A^*)^2 \subset (A^2)^* \subset A^*$ by the Theorem 2.7. (2) is clear.

(3). Since S is regular, 2^S is Hausdorff by Lemma 1.1. Suppose there are two distinct elements x and y in A^* with $xy \neq yx$. Then there exist open subsets $\mathcal{U}$ and $\mathcal{V}$ of 2^S such that $xy \in \mathcal{U}$, $yx \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \square$. Since the multiplication is continuous, there exist open subsets U_x, U_y, V_x, and V_y of S such that $x \in U_x \cap V_x$, $y \in U_y \cap V_y$, $U_x \circ U_y \subset \mathcal{U}$, $V_x \circ V_y \subset \mathcal{V}$. Therefore $((U_x \cap V_x) \circ (U_y \cap V_y)) \cap ((U_y \cap V_y) \circ (U_x \cap V_x)) = \square$. Since there are elements a and b of A such that $a \in U_x \cap V_x$ and $b \in U_y \cap V_y$, $ab \neq ba$. This contradicts the commutativity of A.

Theorem 2.9. If S is regular, then E is closed.

Proof. Suppose there is an element x in $E^* - E$, i.e., $x \in E^*$ and $x \notin x^2$. Since S is regular, there exist open subsets U and V of S such that $x \in U$, $x^2 \subset V$, and $U \cap V = \square$. Since $x^2 \subset V$, by Theorem 2.1, there exist open subsets V_1 and V_2 of S such that $x \in V_1 \cap V_2$ and $V_1 V_2 \subset V$. Therefore $V_1 V_2 \cap U = \square$. Let $W = V_1 \cap V_2 \cap U$. Then $x \in W = W^o$ and $W^2 \cap W = \square$. Since $x \in E^*$, $W \cap E = \square$, i.e., there is an element e in W such that $e \in e^2$. Then $e \in W \cap W^2 \neq \square$, which is a contradiction.

2. Induced Semigroup

Theorem 2.10. Suppose S is compact. Define a multiplication $\cdot$ in 2^S via

$$(\alpha) \qquad A \cdot B = AB$$

for $A, B \in 2^S$. Then $(2^S, \cdot)$ becomes a compact topological semigroup. This new semigroup will be called the induced semigroup of S.

Proof. Let A, B, and C be elements of 2^S. If $x \in (AB)C$, then there is an element y in AB and an element c in C such that $x \in yc$. Since $y \in AB$, there is an element a in A and an element b in B such that $y \in ab$. Then $x \in yc \subset (ab)c = a(bc) \subset A(BC)$, and $(AB)C \subset A(BC)$. Similarly, $(AB)C \supset A(BC)$ holds.

Now let $(A, B) \in 2^S \times 2^S$. Since S is compact, $A \circ B$ is a compact subset of 2^S and hence $AB \in 2^S$ by using Lemma 2.6. If U is an open subset of S such that $A \cdot B \in L(U)$, then $AB \subset U$. By Theorem 2.1, there are open subsets W_1 and W_2 of S such that $A \subset W_1$, $B \subset W_2$, and $W_1W_2 \subset U$. Then $A \in L(W_1)$ and $B \in L(W_2)$. Let $E \in L(W_1)$ and $F \in L(W_2)$. Then $E \subset W_1$ and $F \subset W_2$ and hence $EF \subset W_1W_2 \subset U$. Therefore $L(W_1) \cdot L(W_2) \subset L(U)$. Now let V

be an open subset of S such that $A \cdot B \in M(V)$. Then $AB \cap V \neq \square$. Let $a \in A$ and $b \in B$ such that $ab \cap V \neq \square$. By using Corollary 1.5, there exist open subsets W_3 and W_4 of S such that $a \in W_3$, $b \in W_4$, and $xy \cap V \neq \square$ for each $x \in W_3$ and $y \in W_4$. Therefore, $A \in M(W_3)$ and $B \in M(W_4)$. If $G \in M(W_3)$ and if $H \in M(W_4)$, then $G \cap W_3 \neq \square \neq H \cap W_4$. Let $c \in G \cap W_3$ and let $d \in H \cap W_4$. Since $cd \cap V \neq \square$, $GH \cap V \neq \square$. Hence $M(W_3) \cdot M(W_4) \subset M(V)$, and therefore the multiplication (α) is continuous. Since S is compact and Hausdorff, by Lemma 1.1, 2^S is compact and Hausdorff and hence 2^S is a compact topological semigroup.

Example 2.11. If X is a compact space, then $(2^X, \vee)$ is a topological semilattice under the multiplication

$$(\beta) \qquad A \vee B = A \cup B$$

for $A, B \in 2^X$.

Proof. It may be observed that the operation (β) is the topological semigroup multiplication which is induced by the multi-mob multiplication (1) in Examples 1.7.

Corollary 2.12. If S is compact, then there is a closed subset A of S such that $A^2 = A$.

3. Ideals

Definition 2.13. A nonvoid subset A of S is said to be a left (right, two-sided) ideal of S if and only if $SA \subset A$ ($AS \subset A$, $AS \cup SA \subset A$).

The proofs of the following two theorems are routine, and will be omitted.

Theorem 2.14. Let $A \subset S$ and let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of subsets of S. Then

$$A(\cup\{A_\lambda \mid \lambda \in \Lambda\}) = \cup\{AA_\lambda \mid \lambda \in \Lambda\}, \quad A(\cap\{A_\lambda \mid \lambda \in \Lambda\}) \subset \cap\{AA_\lambda \mid \lambda \in \Lambda\}.$$

Theorem 2.15.

(1) The intersection (if it is nonvoid) of any collection of left (right, two-sided) ideals of S is again a left (right, two-sided) ideal of S.

(2) The closure of a left (right, two-sided) ideal is again a left (right, two-sided) ideal of S.

Theorem 2.16. If S is compact, then each left (right, two-sided) ideal of S contains a minimal left (right, two-sided) ideal which is closed.

Proof. Let L be a left ideal of S and let $\mathfrak{L}$

be the collection of all closed left ideals of S which are contained in L. If $a \in L$, then $Sa \subset SL \subset L$ and $S(Sa) = (SS)a \subset Sa$. It follows that Sa is a left ideal of S contained in L. Since S is compact, by Lemma 2.6, Sa is closed in S and it belongs to $\mathfrak{L}$. Therefore, $\mathfrak{L}$ is nonvoid. $\mathfrak{L}$ is partially ordered by set inclusion. Let $\mathfrak{L}_o$ be a chain in $\mathfrak{L}$. Since $\mathfrak{L}_o$ is a collection of closed subsets of the compact space S with finite intersection property, $\bigcap \mathfrak{L}_o \neq \square$. By Theorem 2.15, $\bigcap \mathfrak{L}_o \in \mathfrak{L}$. Therefore every chain in $\mathfrak{L}$ is lower bounded. By Zorn's lemma, there is a minimal element L_o in $\mathfrak{L}$. Now let L_1 be a left ideal of S which is contained in L_o and let $b \in L_1$. Then Sb is a closed left ideal of S and $Sb \subset L_1 \subset L_o$. Hence $L_1 = L_o$, i.e., L_o is a minimal left ideal of S and is closed. Similar arguments hold for right and two-sided ideals.

<u>Theorem 2.17</u>. The minimal ideal of S is unique.

<u>Proof</u>. Let K_1 and K_2 be minimal ideals of S. Then $K_1 \cap K_2$ is an ideal of S since $\square \neq K_1K_2 \subset K_1 \cap K_2$. Since K_1 and K_2 are minimal, $K_1 = K_1 \cap K_2 = K_2$.

Throughout, K will denote the minimal ideal of S.

Theorem 2.18. Let $\mathcal{L}(\mathcal{R})$ denote the collection of all minimal left (right) ideals of S.

(1) If $\mathcal{L} \neq \square (\mathcal{R} \neq \square)$, then S has the minimal ideal K.

(2) $L_1, L_2 \in \mathcal{L}$ and $L_1 \cap L_2 \neq \square$ imply $L_1 = L_2$.
$R_1, R_2 \in \mathcal{R}$ and $R_1 \cap R_2 \neq \square$ imply $R_1 = R_2$.

(3) $\cup \mathcal{L} \subset K$ and $\cup \mathcal{R} \subset K$.

Proof. (1). Let $L \in \mathcal{L}$ and let I be an ideal of S, then $S(IL) = (SI)L \subset IL$ and hence IL is a left ideal of S. Since $IL \subset SL \subset L \in \mathcal{L}$, $IL = L$. Therefore, $L = IL \subset IS \subset I$, i.e., all minimal left ideals are contained in each ideal of S. Hence $\square \neq \cup \mathcal{L} \subset \cap \{ I \mid IS \cup SI \subset I\} = K$. (2) and (3) are clear.

Remarks. Suppose S is a topological semigroup. Then the minimal ideal of S is the union of all minimal left (right) ideals of S. Moreover, Wallace [22] proved that the minimal ideal of a topological semigroup S is a retract of S. The following examples show that these are not true if S is a multi-mob.

Examples 2.19.

(1). Let S be the multi-mob (5) in Example 1.7, i.e., $S = [a, b]$ and $xy = [a, y]$ for each x and y in S.

In this multi-mob S, $\{a\}$ is the only minimal left ideal of S. On the other hand, the only minimal right ideal of S is S itself. Therefore, $\bigcup\mathfrak{L} = \{a\} \subsetneqq K = S$.

(2). In Example 1.9, the annulus B is the minimal ideal of the multi-mob D. However, it is clear that B is not a retract of D.

Lemma 2.20. Let X be a space and let $\mathfrak{B}$ be a connected subset of 2^X. If there is an element $B \in \mathfrak{B}$ such that B is connected in X, then $\bigcup\mathfrak{B}$ is connected in X [12].

Proof. Let $Y = \bigcup\mathfrak{B}$. Suppose Y is not connected in X. Then there are two disjoint open subsets U and V of X such that $Y \subset U \cup V$, $U \cap Y \neq \square \neq V \cap Y$. Since B is connected in X, $B \subset U$ or $B \subset V$. If $B \subset U$, then $B \in L(U) \cap \mathfrak{B} \neq \square$. Since $Y \cap V \neq \square$, $M(V) \cap \mathfrak{B} \neq \square$. Since $Y \subset U \cup V$, $\mathfrak{B} \subset L(U) \cup M(V)$. Since $U \cap V = \square$, $L(U) \cap M(V) = \square$. Therefore $\mathfrak{B}$ is not connected in 2^X, which is a contradiction.

By routine arguments, one may obtain

Theorem 2.21. Let $L(R, K)$ be a minimal left (right, two-sided) ideal of S. Then $L = Sa(R = aS, K = SaS)$

for each $a \in L (a \in R, a \in K)$.

By the aid of Lemma 2.20 and Theorem 2.21, we have the following:

Theorem 2.22. Let S be connected and let $L(R, K)$ be a minimal left (right, two-sided) ideal of S. Then

(1) If sa is connected for some $s \in S$ and $a \in L$, then L is connected.

(2) If as is connected for some $a \in R$ and $s \in S$, then R is connected.

(3) If sa is connected for some $s \in S$ and $a \in K$ and if rt is connected for some $r \in Sa$ and $t \in S$, then K is connected.

Corollary 2.23. If S is connected, then each minimal left (right, two-sided) ideal is connected when it contains a right (left, two-sided) scalar element.

Remarks. If S is a connected topological semigroup, then it is known that each minimal left (right, two-sided) ideal of S is connected. However, if S is a multi-mob, S may not have this property. Let X be any connected space with more than two elements. Let a and b be two fixed elements of X and define $xy = \{a, b\}$

for each (x, y) in $X \times X$. Then X is a connected multi-mob with its minimal ideal $\{a, b\}$. But $\{a, b\}$ is not connected.

Theorem 2.24. If S is connected and if S has a scalar unit, then each ideal of S is connected.

Proof. Let J be an ideal of S. Since S has a scalar unit, $x \in Sx$ and Sx is connected for each $x \in S$. Since $J = \bigcup\{ x \mid x \in J \} \subset \bigcup\{ Sx \mid x \in J \}$ and since $Sx \subset SJ \subset J$ for each $x \in J$, $J = \bigcup\{ Sx \mid x \in J \}$. Let $y_o \in J$, then $y_oS \subset J$ and hence $J = (\bigcup\{ Sx \mid x \in J \}) \cup y_oS$. Since y_oS is connected and since $y_ox \subset Sx \cap y_oS$ for each $x \in J$, J is connected.

Definition 2.25. For each subset A of S, $J_o(A)$ will denote the union of all ideals of S contained in A. If A contains no ideal of S, then $J_o(A) = \square$. If $J_o(A)$ is nonvoid, then it is clearly the unique largest ideal of S contained in A. $R_o(A)$ and $L_o(A)$ are defined analogously.

Theorem 2.26. Let A be a subset of S.

(1) If A is closed, then $J_o(A)$, $L_o(A)$, and $R_o(A)$ are closed.

(2) If A is open and if S is compact, then $J_o(A)$, $L_o(A)$, and $R_o(A)$ are open.

Proof. (1). Since $J_o(A)$ is an ideal of S, by using Theorem 2.15, $J_o(A)^*$ is an ideal of S. Since $J_o(A)$ is contained in the closed set A, $J_o(A)^* \subset A$ and $J_o(A)^* \subset J_o(A)$. Therefore $J_o(A)^* = J_o(A)$, i.e., $J_o(A)$ is closed.

(2). If $x \in J_o(A)$, then $Sx \subset SJ_o(A) \subset J_o(A) \subset A$. Since S is compact and A is open, by using Theorem 2.1, there exists an open subset U of S such that $x \in U$ and $SU \subset A$. Again, since $xS \subset A$, there is an open subset V of S such that $x \in V$ and $VS \subset A$. Now, since $SxS \subset A$, there is an open subset W of S such that $x \in W$ and $SWS \subset A$. Let $M = U \cap V \cap W \cap A$. Then M is an open subset of S about x. By Theorem 2.14, $M \cup MS \cup SM \cup SMS$ is an ideal of S. Since $M \cup MS \cup SM \cup SMS \subset A \cup VS \cup SU \cup SWS \subset A$, $M \cup MS \cup SM \cup SMS \subset J_o(A)$. Therefore, $J_o(A)$ is open. Similar arguments hold for $L_o(A)$ and for $R_o(A)$.

Theorem 2.27. Suppose S is compact. Then

(1) Each proper ideal of S is contained in a maximal proper ideal of S and each maximal proper ideal is open.

(2) If S is connected, then each maximal proper ideal of S is dense in S.

Proof. (1). Let J be a proper ideal of S and let $a \in S - J$. Since S is compact and $S - \{a\}$ is open, by using Theorem 2.26, $J_o(S - \{a\})$ is a proper open ideal of S containing J. Therefore it is sufficient to consider only open proper ideals. Let $\mathfrak{B}$ be the set of all proper open ideals of S containing J. Then $\mathfrak{B}$ is nonvoid. $\mathfrak{B}$ is partially ordered by set inclusion. Since $J_o(S - \{a\}) \in \mathfrak{B}$, by the Hausdorff Maximal Principle, there exists a maximal chain $\mathcal{C}$ in $\mathfrak{B}$ containing $J_o(S - \{a\})$. Let $M = \bigcup \mathcal{C}$. Then M is a maximal open ideal of S containing J. If M is not proper, i.e., $M = S$, then, $\mathcal{C}$ is an open cover of S. Since S is compact, there exist $M_1, \ldots, M_n \in \mathcal{C}$ such that $M_1 \subset M_2 \subset \ldots \subset M_n$ and $S \subset \bigcup\{ M_j \mid j = 1, \ldots, n \}$, and hence $S = M_n$ which contradicts the fact that M_n is a proper ideal of S. Therefore M is a maximal proper open ideal of S containing J.

(2). Let J be a maximal proper ideal of S. Then J is open. Since S is connected, $J \neq J^*$. Since J^* is also an ideal of S containing J, $J^* = S$.

Definition 2.28. A multi-clan is a continuum

multi-mob with a scalar unit.

Theorem 2.29. Each dense left (right) ideal of a multi-clan containing K is connected, and hence each dense ideal of a multi-clan is connected.

Proof. Let L be a dense left ideal of the multi-clan S containing K. Then $S = S^2 = SL^* \subset (SL)^* \subset L^* = S$ and hence $(SL)^* = S$. Since $SL = S(L \cup K) = SL \cup SK = SL \cup K$, $SL = (\cup\{ Sx \mid x \in K \}) \cup K$. By Theorem 2.22, K and Sx are connected for each $x \in L$. By using Theorem 2.16 and Theorem 2.18, $K \cap Sx \neq \square$ for each $x \in L$, and SL is connected. But then $SL \subset L \subset S = (SL)^*$ implies the connectedness of L.

Theorem 2.30. Suppose S is compact. Let $\bar{L}$ $(\bar{R}, \bar{J})$ be a left (right, two-sided) ideal of the induced semigroup $(2^S, \cdot)$ of S. Then $L = \cup\bar{L}$ $(R = \cup\bar{R},\ J = \cup\bar{J})$ is a left (right, two-sided) ideal of S.

Proof. Since $\bar{L}$ is a left ideal of 2^S, $2^S \cdot \bar{L} \subset \bar{L}$. It follows that $\cup\{ S \cdot A \mid A \in \bar{L} \} \subset \cup 2^S \cdot \bar{L} \subset \cup\bar{L} = L$. By using Theorem 2.14, $\cup\{ S \cdot A \mid A \in \bar{L}\} = \cup\{ SA \mid A \in \bar{L} \} = S(\cup\bar{L}) = SL$. Therefore, $SL \subset L$, and L is a left ideal of S. Similar arguments hold for R and for J.

Theorem 2.31. Suppose S is compact. Let $L(R, J)$ be a closed left (right, two-sided) ideal of S. Then 2^L $(2^R, 2^J)$ is a closed left (right, two-sided) ideal of the induced semigroup $(2^S, \cdot)$ of S.

Proof. Since L is closed, $2^L \subset 2^S$. If $A \in 2^S$ and $B \in 2^L$, then $AB \subset SL \subset L$. Therefore, $2^S \cdot 2^L \subset 2^L$ since AB is closed for each $A \in 2^S$ and each $B \in 2^L$. On account of $2^L = L(L)$, 2^L is a closed left ideal of 2^S.

Theorem 2.32. Suppose S is compact. Let $\bar{K}$ be the minimal ideal of the induced semigroup $(2^S, \cdot)$ of S. Then $K = \cup\bar{K}$.

Proof. In view of Theorem 2.30, K is a closed ideal of S. If K' is the minimal ideal of S, then K' is closed and $K' \subset K$ since K' is unique. By using Theorem 2.31, $2^{K'}$ is a closed ideal of 2^S. Again by the uniqueness of $\bar{K}$, $\bar{K} \subset 2^{K'}$. Therefore $K = \cup\bar{K} \subset \cup 2^{K'} = K'$.

4. Homomorphisms

Definition 2.33. Let S and T be multi-mobs and let $f: S \longrightarrow T$ be a function. Then f is called a *homomorphism* if and only if $f(xy) = f(x)f(y)$ holds for all $x, y \in S$. f is said to be an *isomorphism* if and only if it is a bijective homomorphism. f is called an *iseomorphism* if and only if it is both a homeomorphism and an isomorphism. A continuous homomorphism will be called a *morphism*.

Theorem 2.34. Suppose S and T are multi-mobs and let $A, A' \subset S$, $B, B' \subset T$, and $f: S \longrightarrow T$ be a morphism.

(1) $f(AA') = f(A)f(A')$.

(2) $f^{-1}(B)f^{-1}(B') \subset f^{-1}(BB')$ and equality need not hold even if f is onto.

(3) If A is a submob of S, then $f(A)$ is a submob of T.

If B is a submob of T, then $f^{-1}(B)$ is a submob of S if it is nonvoid.

(4) If A is an ideal of S and if f is onto, then $f(A)$ is an ideal of T.

If B is an ideal of T and if $f^{-1}(B)$ is nonvoid, then $f^{-1}(B)$ is an ideal of S.

If $f^{-1}(B)$ is an ideal of S and if f is onto, then B is an ideal of T.

(5) If f is onto and $K(S)$, the minimal ideal of S, exists, then $f(K(S)) = K(T)$, the minimal ideal of T.

(6) Let $E(S)$ and $E(T)$ be the set of all multi-idempotents of S and T respectively. Then $f(E(S)) \subset E(T)$.

Proof. (1). $f(AA') = f(\bigcup\{aa' \mid a \in A,\ a' \in A'\}) = \bigcup\{f(a)f(a') \mid a \in A,\ a' \in A'\} = f(A)f(A')$.

(2). By using (1), $f(f^{-1}(B)f^{-1}(B')) = ff^{-1}(B)ff^{-1}(B') \subset BB'$, and $f^{-1}(B)f^{-1}(B') \subset f^{-1}(BB')$.

(3). It is clear from (1).

(4). $Tf(A) = f(S)f(A) = f(SA) \subset f(A)$ and $f(A)T \subset f(A)$. $TB \subset B$ and $BT \subset B$ imply $Sf^{-1}(B) = f^{-1}(T)f^{-1}(B) \subset f^{-1}(TB) \subset f^{-1}(B)$ and $f^{-1}(B)S \subset f^{-1}(B)$. $TB = f(S)ff^{-1}(B) = f(Sf^{-1}(B)) \subset ff^{-1}(B) \subset B$ and $BT \subset B$.

(5). By using (4), $f(K(S))$ is an ideal of T. Suppose $f(K(S)) \neq K(T)$, i.e., there is an ideal M of T such that $M \subsetneqq f(K(S))$. Since f is onto, $ff^{-1}(M) = M \subsetneqq f(K(S))$ and hence $f^{-1}(M) \subsetneqq K(S)$, which is a contradiction.

(6). If $a \in f(E(S))$, then $a = f(e)$ for some $e \in E(S)$.

Since $e \in e^2$, $a^2 = f(e)^2 = f(e^2)$ and $a = f(e) \in f(e^2)$. Therefore, $a \in a^2$, i.e., $f(E(S)) \subset E(T)$.

Theorem 2.35. Let S and T be compact multi-mobs and let $f: S \longrightarrow T$ be a morphism. Then $f^*: 2^S \longrightarrow 2^T$, via $f^*(A) = f(A)$, is a morphism.

Proof. Let $A, B \in 2^S$. By using (1) in Theorem 2.34, $f^*(AB) = f(AB) = f(A)f(B) = f^*(A)f^*(B)$, i.e., f^* is a homomorphism. It will be shown that f^* is continuous.

Let V be an open subset of T such that $f^*(A) \in L(V)$. Then $f(A) \subset V$ and hence $f(a) \in V$ for each $a \in A$. Since f is continuous, there is an open subset U_a of S such that $a \in U_a$ and $f(U_a) \subset V$ for each $a \in A$. Let $U = \bigcup\{ U_a \mid a \in A \}$. Then U is open in S and $A \in L(U)$. If $B \in L(U)$, then $B \subset U$ and $f^*(B) = f(B) \subset f(U) = \bigcup\{ f(U_a) \mid a \in A \} \subset V$, i.e., $f^*(B) \in L(V)$. Therefore, $f^*(L(U)) \subset L(V)$.

Now let W be an open subset of T such that $f^*(A) \in M(W)$, i.e., $f(A) \cap W \neq \square$. Let $a \in A$ such that $f(a) \in W$. Since f is continuous, there is an open subset U' of S such that $a \in U'$ and $f(U') \subset W$. Since $a \in A \cap U' \neq \square$, $A \in M(U')$. Let $C \in M(U')$. Then $C \cap U' \neq \square$ and hence there is an element $b \in C$ such that $f(b) \in W$.

Therefore, $f(C) \cap W \neq \square$, i.e., $f^*(M(U')) \subset M(W)$, so that f^* is continuous.

Definition 2.36.

(1) $\mathcal{E} \subset S \times S$ is called a multi-congruence on S if and only if it is an equivalence relation and, for each $(x, y), (x', y') \in \mathcal{E}$, the following hold:

$$M(xx') \cap S/\mathcal{E} = M(yy') \cap S/\mathcal{E}.$$

(2) If $f: X \longrightarrow Y$, then the kernel of f is defined to be the set

$$\text{Ker } f = \{(x, y) \in X \times X \mid f(x) = f(y)\}.$$

Let $\mathcal{E}$ be an equivalence relation on the space X. Let $\mathcal{E}(x) = p((X \times \{x\}) \cap \mathcal{E})$, where p is the first projection. Then $\{\mathcal{E}(x) \mid x \in X\}$ is a disjointed cover of X and $X/\mathcal{E} = \{\mathcal{E}(x) \mid x \in X\}$. If X is compact and $\mathcal{E}$ is closed, then $X/\mathcal{E} \subset 2^X$ since the projection p is closed. It is well known that $X/\mathcal{E}$ is a compact Hausdorff space [6] under the quotient topology if X is compact and $\mathcal{E}$ is closed. Since $X/\mathcal{E}$ is a subset of 2^X when X compact and $\mathcal{E}$ closed, it is natural to observe that the relation between the quotient topology on $X/\mathcal{E}$ and the relative topology on $X/\mathcal{E}$ induced by the Vietoris topology of 2^X.

Theorem 2.37. If $\mathcal{E}$ is a closed equivalence relation on the compact space X and if $X/\mathcal{E}$ is a closed subset of 2^X, then the quotient topology on $X/\mathcal{E}$ is equivalent to the relative topology induced on $X/\mathcal{E}$ by the Vietoris topology of 2^X.

Proof. Let $\mathcal{J}$ be a closed subset of $X/\mathcal{E}$ relative to the quotient topology of $X/\mathcal{E}$. Then $G = \bigcup \mathcal{J}$ is closed in X and hence $L(G)$ is closed in 2^X. Therefore, $\mathcal{J}$ is closed in $X/\mathcal{E}$ with respect to the relative topology of the Vietoris topology of 2^X.

Now let $\mathcal{H}$ be a closed subset of $X/\mathcal{E}$ with respect to the relative topology of 2^X. Since $X/\mathcal{E}$ is closed in 2^X, $\mathcal{H}$ is compact in 2^X, and $H = \bigcup \mathcal{H}$ is closed in X. Therefore, $\mathcal{H}$ is closed in $X/\mathcal{E}$ with respect to the quotient topology of $X/\mathcal{E}$.

Theorem 2.38. Let S be compact and let $\mathcal{E}$ be a closed equivalence relation on S such that $S/\mathcal{E}$ is closed in 2^S. Let $\sigma\colon 2^S \longrightarrow 2^{S/\mathcal{E}}$ be a function defined by $\sigma(A) = M(A) \cap S/\mathcal{E}$. Then σ is continuous.

Proof. Let $\mathfrak{B} \subset S/\mathcal{E}$. It will be shown that $\{A \in 2^S \mid \sigma(A) \cap \mathfrak{B} \neq \square\} = M(\bigcup \mathfrak{B})$. If $A \in 2^S$ such that $\sigma(A) \cap \mathfrak{B} \neq \square$, then $A \cap B \neq \square$ for some B in $\mathfrak{B}$, and

$A \cap (\bigcup \mathfrak{B}) \neq \square$. Therefore, $A \in M(\bigcup \mathfrak{B})$. Conversely, if A is an element of $M(\bigcup \mathfrak{B})$, then $A \cap (\bigcup \mathfrak{B}) \neq \square$. It follows that $B \in M(A)$ for some $B \in \mathfrak{B}$. This completes the proof of the above assertion. Now by using Theorem 2.37, $\mathfrak{B}$ is open (closed) in $S/\mathcal{E}$ if and only if $\bigcup \mathfrak{B}$ is open (closed) in S. Therefore, $M(\bigcup \mathfrak{B})$ is open (closed) in 2^S if $\bigcup \mathfrak{B}$ is open (closed) in S. By using Lemma 1.3, σ is continuous.

Lemma 2.39. Let $f: X \longrightarrow Y$ and $g: X \longrightarrow Z$ be continuous with $\operatorname{Ker} g \subset \operatorname{Ker} f$. Suppose that g is onto and that V is open if and only if $g^{-1}(V)$ is open in X. Then there exists a unique continuous function h making the following diagram analytic.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Corollary 2.40. Suppose $f: X \longrightarrow Y$ is continuous. Let $\mathcal{E}$ be an equivalence relation on X and let q be the quotient map from X onto $X/\mathcal{E}$. If $\operatorname{Ker} q \subset \operatorname{Ker} f$, then there exists a unique continuous function h making the

following diagram analytic.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\mathcal{E} \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Corollary 2.41. (Sierpinski). Suppose $f: X \longrightarrow Y$ and $g: X \longrightarrow Z$ are continuous with $\operatorname{Ker} g \subset \operatorname{Ker} f$. If g is onto and if X is compact, then there exists a unique continuous function h making the diagram in Lemma 2.39 analytic. Furthermore, if f is onto and $\operatorname{Ker} g = \operatorname{Ker} f$ then h is a homeomorphism. If f and g are homomorphisms of multi-mobs, then h is also.

Theorem 2.42. Suppose S is compact. If $\mathcal{E}$ is a closed multi-congruence on S such that $S/\mathcal{E}$ is closed in 2^S, then there exists a unique continuous function θ making the following diagram analytic, where q is the quotient map and m is the multi-mob multiplication on S.

$$\begin{tikzcd} S \times S \arrow[r, "m"] \arrow[d, "q \times q"'] & 2^S \arrow[d, "\sigma"] \\ S/\mathcal{E} \times S/\mathcal{E} \arrow[r, dashed, "\theta"'] & 2^{S/\mathcal{E}} \end{tikzcd}$$

Proof. Let $\lambda = \sigma m$. Then λ is continuous by Theorem 2.38. By the aid of Corollary 2.41, it is only required to show that $\mathrm{Ker}\,(q \times q) \subset \mathrm{Ker}\,\lambda$. Now let $((x, y), (x', y')) \in \mathrm{Ker}\,(q \times q)$. Then $q(x) = q(x')$ and $q(Y) = q(y')$, i.e., $(x, x'), (y, y') \in \mathrm{Ker}\, q = \mathcal{E}$. Since $\mathcal{E}$ is a multi-congruence on S, $M(xy) \cap S/\mathcal{E} = M(x'y') \cap S/\mathcal{E}$, i.e., $\sigma m(x, y) = \sigma m(x', y')$. It follows that $\lambda(x, y) = \lambda(x', y')$, i.e., $\mathrm{Ker}\,(q \times q) \subset \mathrm{Ker}\,\lambda$.

Theorem 2.43. Suppose S is compact. If $\mathcal{E}$ is a multi-congruence on S such that $S/\mathcal{E}$ is closed in 2^S, then $(S/\mathcal{E}, \wedge)$ is a multi-mob, via $q(x) \wedge q(y) = M(xy) \cap S/\mathcal{E}$, and the quotient map q is a morphism.

Proof. For the proof of the first part, it is only required to show that the operation $\wedge$ is associative. If $u, v \in S$, then $q(u) \wedge q(v) = M(uv) \cap S/\mathcal{E}$ $= \{ q(w) \mid w \in uv \}$. If $x, y, z \in S$, then $(q(x) \wedge q(y)) \wedge q(z) = \bigcup \{ q(a) \wedge q(z) \mid a \in xy \}$. Claim: $(q(x) \wedge q(y)) \wedge q(z)$ $= \{ q(c) \mid c \in xyz \}$.

If $a \in xy$, then $q(a) \wedge q(z) = \{ q(b) \mid b \in az \}$ $\subset \{ q(c) \mid c \in xyz \}$. Therefore, $(q(x) \wedge q(y)) \wedge q(z) \subset \{ q(c) \mid c \in xyz \}$.

If $c \in xyz$, then $c \in az$ for some $a \in xy$, i.e.,

$q(c) \in q(a) \wedge q(z)$ for some $a \in xy$. It follows that $q(c) \in \bigcup \{ q(a) \wedge q(z) \mid a \in xy \} = (q(x) \wedge q(y)) \wedge q(z)$, i.e., $\{ q(c) \mid c \in xyz \} \subset (q(x) \wedge q(y)) \wedge q(z)$. Now the associativity is clear.

Then, since $(S/\mathcal{E}, \wedge)$ is a multi-mob, the homomorphism between S and $S/\mathcal{E}$ may be considered. If $x, y \in S$, then $q(xy) = \{ q(b) \mid b \in xy \} = q(x) \wedge q(y)$ and hence q is a morphism.

Theorem 2.44. Let S and T be multi-mobs. If $f\colon S \longrightarrow T$ is a morphism, then $\operatorname{Ker} f$ is a closed multi-congruence on S.

Proof. Since $\operatorname{Ker} f = (f \times f)^{-1}(\Delta)$, where Δ is the diagonal relation on T, $\operatorname{Ker} f$ is a closed equivalence relation on S. If $(x, y), (x', y') \in \operatorname{Ker} f$, then $f(x) = f(y)$ and $f(x') = f(y')$. Since f is a homomorphism, $f(xx') = f(yy')$. Now let $A \in M(xx') \cap S/\operatorname{Ker} f$. Since $A \in M(xx')$, there is an element t in $A \cap xx'$. Then $f(t) \in f(xx')$, and $f(t) = f(s)$ for some $s \in yy'$, i.e., $(t, s) \in \operatorname{Ker} f$. Since $t \in A \in S/\operatorname{Ker} f$, it follows that $s \in A \cap yy' \neq \Box$. Therefore, $A \in M(yy') \cap S/\operatorname{Ker} f$, i.e.,

$$M(xx') \cap S/\operatorname{Ker} f \subset M(yy') \cap S/\operatorname{Ker} f.$$

Similarly the other inclusion could be shown.

Theorem 2.45. Let S and T be multi-mobs. If $f: S \longrightarrow T$ is an onto morphism such that $S/\mathrm{Ker}\, f$ is closed in 2^S, then there is a unique iseomorphism $\bar{f}$: $S/\mathrm{Ker}\, f \longrightarrow T$ making the following diagram analytic.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "q"'] & T \\
S/\mathrm{Ker}\, f \arrow[ur, dashed, "\bar{f}"'] &
\end{tikzcd}$$

Proof. This is an immediate consequence of the above two theorems and Sierpinski's Lemma (Corollary 2.41).

Theorem 2.46. Suppose S is compact and let J be a closed ideal of S. Then

(1) $\mathcal{E} = J \times J \cup \Delta$ is a closed multi-congruence on S.

(2) If $S/\mathcal{E}$ is closed in 2^S, then $(S/\mathcal{E}, \wedge)$ is a multi-mob having J as a zero.

Proof. (1). Clearly $\mathcal{E}$ is a closed equivalence relation on S. Let $(x, y), (x', y') \in \mathcal{E}$ and let $A \in M(xx') \cap S/\mathcal{E}$. Since $A \in M(xx')$, there is an element

t in $A \cap xx'$.

If $(x, y), (x', y') \in \Delta$, then $A \in M(yy')$.

If $(x, y) \in J \times J$ or $(x', y') \in J \times J$, then in either case, $xx', yy' \subset J$ since J is an ideal of S. Since $t \in xx'$, $(t, z) \in J \times J$ for all $z \in yy'$. Again since $t \in A \in S/\mathcal{E}$, $yy' \subset A$ and $A \in M(yy') \cap S/\mathcal{E}$. Therefore, $M(xx') \cap S/\mathcal{E} \subset M(yy') \cap S/\mathcal{E}$. The other containment is obtained in a similar way.

(2). If $x \in J$, $y \in S$, then $q(x) = J$ and $xy \subset J$. Hence, $q(x) \wedge q(y) = \{ q(a) \mid a \in xy \} = J = q(y) \wedge q(x)$.

We shall write $S/[J]$ rather than $S/(J \times J) \cup \Delta$ and $S/[J]$ will be called the <u>Rees quotient</u> of S.

CHAPTER III

MULTI-MOBS ON AN INTERVAL

The purpose of this chapter is to investigate some properties of multi-mobs on a real closed interval. Throughout this chapter, I will denote the real closed interval from a to b and a semigroup will always mean a topological semigroup.

In chapter I, it was shown that a multi-mob has a scalar idempotent whenever it has a scalar element. It seems that, as far as multi-mobs are concerned, the role of idempotents is important. In [17], W. L. Strother has proven that I has the fixed point property for multi-valued functions, i.e., there is a point p in I such that $p \in f(p)$ for any continuous function $f: I \longrightarrow 2^I$. Since the function $f: I \longrightarrow 2^I$, via $f(x) = x^2$, is continuous whenever I is a multi-mob, one may obtain

Theorem 3.1. If I is a multi-mob, then I has a multi-idempotent. (see also [11] and [25])

1. Standard Multi-Mob

The following theorem, for a semigroup, is due to A.D. Wallace and J. M. Day and appears in [2], [3], and [4]. By using Theorem 2.3, the same theorem holds for a multi-mob. Because of the importance and frequent use of this theorem, the proof will be presented here in spite of its similarities.

Theorem 3.2. Suppose S is a continuum multi-mob. If H is a subset of S with nonempty boundary $F(H)$ and if H^* contains a point a such that $Sa \subset H^*$, then $Sb \subset H^*$ for some b in $F(H)$.

Proof. By the hypothesis of this theorem, $H^* \cap S^{[-1]}H^*$ is nonvoid since it contains a. Let G be a component of $H^* \cap S^{[-1]}H^*$. Then $G \cup SG \subset H^*$ and, therefore, $G^* \cup SG^* \subset G^* \cup (SG)^* = (G \cup SG)^* \subset H^*$. If $(G^* \cup SG^*) \cap F(H) = \square$, then $G^* \cup SG^* \subset H^o$, i.e., $G^* \subset H^o \cap S^{[-1]}H^o \subset H^* \cap S^{[-1]}H^*$. It follows that $G^* = G$, and G is a component of the proper open subset $H^o \cap S^{[-1]}H^o$. Therefore, one may obtain that $G^* \cap [(H^o \cap S^{[-1]}H^o)^* - (H^o \cap S^{[-1]}H^o)] \neq \square$ (see [21]). This is a contradiction. Hence there is a point b in $(G^* \cup SG^*) \cap F(H)$. Then $Sb \subset S(G^* \cup SG^*) = SG^* \cup S^2G^* \subset SG^* \subset H^*$.

As an immediate application to this theorem and Lemma 2.20, we have

Corollary 3.3. Suppose I is a multi-mob in which a is a zero.

(1) $[a, x]$ is an ideal of I for each x in I.

(2) If I has a unit, then $Ix = [a, x] = xI$ for each x in I.

(3) If e is a multi-idempotent of I, then $Ie = [a, e] = eI$.

Lemma 3.4. If $f(g)\colon 2^I \longrightarrow I$ is a function defined by $f(A) = \inf A$ ($g(A) = \sup A$), then $f(g)$ is continuous.

Proof. Let $A \in 2^I$ and let $U = (c, d)$, the open interval from c to d, such that $f(A) = x \in U$. Let $V = (c, b]$. Then $A \subset U \cup V$ and $x \in A \cap V \neq \square$, i.e., $A \in \langle U, V\rangle$. If $B \in \langle U, V\rangle$, then $B \subset V$ and $B \cap U \neq \square$, and, therefore, $c < f(B) < d$, i.e., $f(B) \in U$. It follows that $f(\langle U, V\rangle) \subset U$, and f is continuous. Similarly, g is continuous.

For a general version of this lemma, one might look at E. Michael [12].

Definition 3.5. A multi-mob on I will be called a standard multi-mob if and only if a is a zero and b is a scalar unit. For the definition of a standard thread in semigroup theory, see [1], [14], [16], and [20].

Lemma 3.6. Suppose I is a multi-mob in which a is a zero. If e is an idempotent of I, then $[a, e]$ is a standard multi-mob. In particular, I is a standard multi-mob if b is an idempotent.

Proof. Suppose e is an idempotent of I. For each x in $[a, e]$, define $x' = \inf(ex)$. In view of (1) in Corollary 3.3, $(x')' \leq x' \leq x$ for each x in $[a, e]$. By the definition of a multi-mob, ex is a closed subset of I so that $x' \in ex$ for each x in $[a, e]$. Since e is an idempotent, $ex = e^2x = e(ex) = \bigcup\{ ey \mid y \in ex \}$ and, therefore, $ey \subset ex$ for each $y \in ex$. It follows that $ex' \subset ex$, and $(x')' = \inf(ex') \geq \inf(ex) = x' \geq (x')'$, i.e., $(x')' = x'$. Now define a function $f\colon [a, e] \longrightarrow [a, e]$, via $f(x) = x'$. Then f is continuous by using Lemma 3.4. Moreover, $f^2(x) = f(f(x)) = f(x') = (x')' = x' = f(x)$, i.e., $f^2 = f$ and f is a retraction. Since $f(a) = a' = a$ and $f(e) = e' = e$, f is an onto function. Hence $f(x) = x$, i.e., $ex = x$ for each x in $[a, e]$.

Similarly, $xe = x$ for each x in $[a, e]$ so that e is a scalar unit for $[a, e]$. By using (i) in Corollary 3.3, $[a, e]$ is a submob of I. Consequently, $[a, e]$ is a standard multi-mob.

<u>Conventions</u>. For a multi-mob on I, the following notation will be adopted throughout the remainder of this chapter. For each x and each y in I, denote

$$x \wedge y = \inf(xy), \qquad x \vee y = \sup(xy).$$

<u>Lemma 3.7</u>. Let I be a standard multi-mob and let $x, y, u, v \in I$ with $x \leq y$ and $u \leq v$. Then

$$x \vee u \leq y \vee v.$$

<u>Proof</u>. Since $x \leq y$, $x \in [a, y] = Sy$ by Corollary 3.3. Then $xu \subset (Sy)u = S(yu) = \bigcup\{ St \mid t \in yu \} = \bigcup\{ [a, t] \mid t \in yu \} = [a, y \vee u]$. It follows that $x \vee u \leq y \vee u$. In a similar way, $y \vee u \leq y \vee v$ may be established. Therefore, $x \vee u \leq y \vee u \leq y \vee v$.

<u>Theorem 3.8</u>. If I is a standard multi-mob, then $(I, \vee)$ is a standard thread.

<u>Proof</u>. In view of Lemma 3.4, the operation $\vee$ is continuous. It only remains to show the associativity of

the multiplication $\vee$. Let $x, y, z \in I$. Since $x \vee y \in xy$, $(x \vee y) z \subset (xy) z = xyz$, i.e.,

$$(x \vee y) \vee z \le \sup(xyz) = \sup(\bigcup\{ tz \mid t \in xy \}) = \sup\{ t \vee z \mid t \in xy \}.$$

Since $t \le x \vee y$ for every $t \in xy$, by using Lemma 3.7, $t \vee z \le (x \vee y) \vee z$ for all t in xy. It follows that $(x \vee y) \vee z \le \sup(xyz) = \sup\{ t \vee z \mid t \in xy \} \le (x \vee y) \vee z$, and $(x \vee y) \vee z = \sup(xyz)$. Similarly, $x \vee (y \vee z) = \sup(xyz)$, i.e., $(x \vee y) \vee z = \sup(xyz) = x \vee (y \vee z)$.

<u>Lemma 3.9</u>. Let I be a standard multi-mob such that $x \wedge z \ne y \wedge z$ for all $x, y, z \in I$ with $x < y$ and $z \ne a$. Then $x \le y$ implies $x \wedge z \le y \wedge z$ for all $z \in I$.

<u>Proof</u>. Let $u < v$ in I and let $A = \{ z \in (a, b] \mid u \wedge z < v \wedge z \}$. Then $A \ne \Box$ since $b \in A$. If $z_o \in A$, then $u \wedge z_o < v \wedge z_o$. Pick a point t so that $u \wedge z_o < t < v \wedge z_o$. By the continuity of the operation $\wedge$, there is an open set W about z_o such that $\{ u \wedge w \mid w \in W \} \subset [a, t)$ and $\{ v \wedge w \mid w \in W \} \subset (t, b]$, i.e., $W \subset A$. Therefore, A is an open subset of $(a, b]$. By the hypothesis of this lemma, $(a, b] - A = \{ z \in (a, b] \mid u \wedge z > v \wedge z \}$. In a similar way, it can be also shown that $(a, b] - A$ is open. Then

A is a proper clopen (closed and open) subset of $(a, b]$ if $(a, b] - A$ is nonvoid. This is a contradiction, and $A = (a, b]$. Therefore, $u \leq v$ implies $u \wedge w \leq v \wedge w$ for all w in I.

Theorem 3.10. Suppose I is a standard multi-mob such that $x \wedge z \neq y \wedge z$ for all $x, y, z \in I$ with $x < y$ and $z \neq a$. Then $(I, \wedge)$ is a standard thread.

Proof. In view of Lemma 3.4, the operation $\wedge$ is continuous. Let $x, y, z \in I$. Since $x \wedge y \in xy$, $(x \wedge y) z \subset (xy) z = xyz$. Hence $(x \wedge y) \wedge z \geq \inf (xyz) = \inf (\bigcup \{ tz \mid t \in xy \}) = \inf \{ t \wedge z \mid t \in xy \}$. Since $t \geq x \wedge y$ for every $t \in xy$, by using Lemma 3.9, $t \wedge z \geq (x \wedge y) \wedge z$ for all t in xy. It follows that $(x \wedge y) \wedge z \geq \inf (xyz) = \inf \{ t \wedge z \mid t \in xy \} \geq (x \wedge y) \wedge z$, i.e., $(x \wedge y) \wedge z = \inf (xyz)$. Similarly, $x \wedge (y \wedge z) = \inf (xyz)$.

Theorem 3.11. Suppose $(I, *)$ and $(I, *')$ are standard threads such that $x * y \leq x *' y$ for each $x, y \in I$. Then I is a standard multi-mob under the multiplication (denoted by juxtaposition) $xy = [x * y, x *' y]$.

Proof. Clearly the multiplication is continuous. To show the distributive law, let x, y, z be in I.

Since $*$ and $*'$ are standard thread multiplications, $t_1 * z \leq t_2 * z$ and $t_1 *' z \leq t_2 *' z$ whenever $t_1 \leq t_2$. If $t \in xy$ then $x * y \leq t \leq x *' y$ so that $(x * y) * z \leq t * z$ and $t *' z \leq (x *' y) *' z$. Since tz is connected for all t in xy, by using Lemma 2.20, $(xy)z$ is connected. It follows that $(xy)z = [(x * y) * z, (x *' y) *' z] = [x * (y * z), x *' (y *' z)] = x(yz)$, i.e., $(xy)z = x(yz)$. Clearly, $ax = a = xa$ and $bx = x = xb$.

<u>Example 3.12</u>. For $x, y \in I$, let $x * y = x \cdot y$, the usual product of the real numbers x and y, and let $x *' y = \min\{x, y\}$. Then I is a standard multi-mob under the multiplication $xy = [x * y, x *' y]$.

<u>Problem A</u>. The only standard multi-mob multiplication on I, in which xy is connected for all $x, y \in I$, has the form $xy = [x * y, x *' y]$, where $*$ and $*'$ are standard thread multiplications on I with $x * y \leq x *' y$ for each $x, y \in I$.

<u>Remark</u>. If one could remove the hypothesis of Theorem 3.10, then Problem A remains true and the structure of the standard multi-mobs on I, in which xy is connected for all $x, y \in I$, is completely determined.

2. Multi-Semilattice

Definition 3.13. A multi-mob S is said to be a multi-band if and only if every element is an idempotent.

A multi-semilattice is a commutative multi-band.

Theorem 3.14. If I is a multi-band in which a is a zero, then $xy = \min\{x, y\}$, i.e., each such multi-band is a topological semilattice.

Proof. Since every element is an idempotent, by using Lemma 3.6, it is readily shown that $xy = x = yx$ whenever $x \leq y$, i.e., $xy = \min\{x, y\}$.

Lemma 3.15. Suppose I is a multi-band. If $v \in uv$ $(u \in uv)$ for all u and v in I with $u < v$, then $uv \cap (v, b] = \square$ $(uv \cap [a, u) = \square)$.

Proof. Let u and v be in I with $u < v$. For each $x \in [u, b]$, let $u \vee x = x'$. By hypothesis, $x \leq x'$. Since $x' \in [u, b]$, $x' \leq (x')'$. Since ux is closed for each x, $ux' \subset ux$. Then $(x')' \leq x'$ so that $(x')' = x'$. Let $A = \{ ux \mid x \in [u, b]\}$. Define the functions $f\colon [u, b] \longrightarrow A$ and $g\colon A \longrightarrow [u, b]$, via $f(x) = ux$ and $g(ux) = x'$, then $h = g \circ f$ is continuous and $h^2 = h$.

Since $h(u) = u$ and $h(b) = b$, h is an onto function. It follows that $h(x) = x$ for all $x \in [u, b]$, and hence $u \vee v = v$, i.e., $uv \cap (v, b] = \square$.

As an immediate consequence to the above lemma, one may obtain the following:

Theorem 3.16. If I is a multi-band such that $x, y \in xy$ and $xy \cap (x, y) = \square$ for each $x, y \in I$ with $x < y$, then I is a multi-semilattice and $xy = \{x, y\}$.

Theorem 3.17. If I is a multi-band such that $xy \cap (x, y) \neq \square$ for each $x, y \in I$ with $x < y$, then I is a multi-semilattice and $xy = [x, y]$.

Proof. Let $x, y \in I$ with $x < y$. Suppose $[x, y] - xy \neq \square$ and let $z \in [x, y] - xy$. Since z is in the open set $I - xy$, let (c, d) be the component containing z in $I - xy$. Since xy is closed, $c, d \in xy$. Then $cd \subset xy$, and $cd \cap (c, d) = \square$. This is a contradiction. Therefore, $[x, y] \subset xy$. Since $x, y \in xy$ for each $x, y \in I$, by using Theorem 3.15, $xy = [x, y]$.

In the following, some multi-semilattice operations on I, other than those that have been given, may be found.

Examples 3.18. Let $a < c < b$.

(1). Define a multiplication on I as follows:

$$xy = yx = \begin{cases} [x, y] & (x, y \in [a, c],\ x \le y) \\ \{x, y\} & (x, y \in [c, b]) \\ [x, c] \cup \{y\} & (x \in [a, c],\ y \in [c, b]). \end{cases}$$

Then I is a multi-semilattice under this multiplication.

Proof. In view of Proposition 17 in [18], the proof of the continuity of this multiplication is routine. Let $x \in [a, c]$ and let $y_1, y_2 \in [c, b]$ with $y_1 \le y_2$. Then $(xy_1)y_2 = [x, c] \cup \{y_1, y_2\} = x(y_1y_2)$. Now let $x_1, x_2 \in [a, c]$ with $x_1 \le x_2$ and let $y \in [c, b]$. Then $x_1(x_2y) = [x_1, c] \cup \{y\} = (x_1x_2)y$.

(2). I is a multi-semilattice under the multiplication

$$xy = yx = \begin{cases} [x, y] & (x, y \in [a, c],\ x \le y) \\ \min\{x, y\} & (x, y \in [c, b]) \\ [x, c] & (x \in [a, c],\ y \in [c, b]). \end{cases}$$

(3). I is also a multi-semilattice under the multiplication

$$xy = yx = \begin{cases} \{x, y\} & (x, y \in [a, c]) \\ \min\{x, y\} & (x, y \in [c, b]) \\ \{x, c\} & (x \in [a, c],\ y \in [c, b]). \end{cases}$$

Problem B. What kinds of multi-semilattice multiplications does I admit?

Problem C. Does a circle C admit a multi-mean structure (a continuous function $m: C \times C \longrightarrow 2^C$ such that $m(x, x) = x$ and $m(x, y) = m(y, x)$ for all $x, y \in C$) except the trivial one?

Remarks. It is well known that a circle C does not admit a mean structure. However, any space X admits a trivial multi-mean structure, $xy = \{x, y\}$, for all $x, y \in X$. In this section, some of the multi-semilattice multiplications were discussed and answered Problem B partially. Problems B and C were posed by Professor A. D. Wallace and remain unsolved.

CHAPTER IV

MULTI-ACTION

1. Multi-Action

Definition 4.1. A multi-action is such a continuous function

$$\mu : S \times X \longrightarrow 2^X$$

that S is a multi-mob and X is a nonvoid Hausdorff space and, denoting $\mu(s, x) = sx$, the associativity condition $(s_1 s_2)x = s_1(s_2 x)$ holds for all $s_1, s_2 \in S$ and all $x \in X$, where $MA = \cup\{ sx \mid s \in M, x \in A \}$ for each $M \subset S$ and each $A \subset X$. (see [15] and [24] for the definition of a semigroup action) We shall say S multi-acts on X via the multi-action μ. If $SX = X$, then μ will be called an onto multi-action ([3], [15]).

Conventions. Throughout this chapter, S and T will denote multi-mobs. Also, we shall adopt the notations $M^{[-1]}A$ and $M^{(-1)}A$ analogous to those in chapter II.

A few theorems below are similar to those in chapter II and will be stated without their proofs.

Theorem 4.2. Suppose T multi-acts on a space X. Let $M \subset T$ and $A \subset X$.

(1) If M and A are compact and if $MA \subset W$ for some open subset W of X, then there exist an open subset U of T and an open subset V of X such that $M \subset U$, $A \subset V$, $UV \subset W$.

(2) If X is regular, then $M^*A^* \subset (MA)^*$ with $M^*A^* = (MA)^*$ whenever A^* and M^* are compact.

(3) i) If M is compact and if A is open, then $M^{[-1]}A$ is open.

ii) If M is compact and if A is closed, then $M^{(-1)}A$ is closed.

iii) If A is compact, then $\{x \in X \mid A \subset Mx\}$ is closed.

Theorem 4.3. Let T be compact and let T multi-act on a continuum X. If H is a subset of X with non-empty boundary $F(H)$ and if H^* contains a point a such that $Ta \subset H^*$, then $Tb \subset H^*$ for some $b \in F(H)$.

Definition 4.4. Let T multi-act on a space X. A nonvoid subset A of X is called a T-ideal of X if

and only if $TA \subset A$. $J_o(A)$ is defined analogously as before.

<u>Theorem 4.5</u>. Suppose T multi-acts on a space X.

(1) The intersection (if it is nonvoid) and the union of T-ideals of X is again a T-ideal of X.

(2) The closure of a T-ideal of X is a T-ideal of X.

(3) If T and X are compact, then each T-ideal of X contains a minimal T-ideal of X and each such is closed.

<u>Theorem 4.6</u>. Let T multi-act on a space X and let A be a subset of X.

(1) If A is closed, then $J_o(A)$ is closed. If A is open and if T is compact, then $J_o(A)$ is open.

(2) If T and X are compact and if A is a proper T-ideal of X, then X contains a proper T-ideal of X containing A which is maximal. Moreover, each maximal proper T-ideal of X is open.

(3) If T is compact and if X is a continuum, then each maximal proper T-ideal of X is open and dense.

<u>Theorem 4.7</u>. Suppose T multi-acts on a space X

with T and X compact. If the multi-action is onto and if $Tx \subset Ty$ or $Ty \subset Tx$ for each $x, y \in X$, then $Tb = X$ for some b in X [15].

2. Induced Action

Suppose T multi-acts on a space X, via μ, with T and X compact. Let

$$\mu^* : 2^T \times 2^X \longrightarrow 2^X$$

be a function defined by $\mu^*(M, A) = MA$. On account of Theorem 4.1, μ^* is a semigroup action. The proof is similar to that of Theorem 2.11. The semigroup action μ^* will be called the <u>induced action</u> of the multi-action μ.

<u>Lemma 4.8</u>. Suppose a semigroup G acts on a space X, via μ, with G and X compact. Let H be a semigroup, Y be a nonvoid Hausdorff space, f be a morphism of G onto H, and g be a continuous function of X onto Y. If ν is any function from $H \times Y$ into Y making the following diagram analytic, then ν is an action [15].

$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "f \times g"'] & X \arrow[d, "g"] \\ H \times Y \arrow[r, "\nu"'] & Y \end{tikzcd}$$

<u>Theorem 4.9</u>. Suppose T multi-acts on a space X,

via μ, with T and X compact. Let Y be a space, f be a morphism of T onto S, and g be a continuous function of X onto Y. If ν is any function from $S \times Y$ into 2^Y making the following diagram analytic then ν is a multi-action, where $g^*(A) = g(A)$.

$$\begin{tikzcd} T \times X \arrow[r, "\mu"] \arrow[d, "f \times g"'] & 2^X \arrow[d, "g^*"] \\ S \times Y \arrow[r, "\nu"'] & 2^Y \end{tikzcd}$$

Proof. In view of Theorem 2.35, $f^*: 2^T \longrightarrow 2^S$, via $f^*(M) = f(M)$, is an onto morphism. Since g is onto, Y is regular and hence g^* is continuous and onto.

Let ν^* be the function defined on $2^S \times 2^Y$ by $\nu^*(N, B) = \cup\{ \nu(n, b) \mid n \in N, b \in B \}$. Now let $(M, A) \in 2^T \times 2^X$. Then

$$\begin{aligned} \nu^*(f^* \times g^*)(M, A) &= \nu^*(f^*(M), g^*(A)) \\ &= \cup\{ \nu(f(m), g(a)) \mid m \in M, a \in A \} \\ &= \cup\{ \nu(f \times g)(m, a) \mid m \in M, a \in A \} \\ &= \cup\{ g^*\mu(m, a) \mid m \in M, a \in A \} \end{aligned}$$

$$= \bigcup \{ g(ma) \mid m \in M,\ a \in A \}$$

$$= g(\bigcup \{ ma \mid m \in M,\ a \in A \})$$

$$= g(MA) = g^*(MA) = g^*\mu^*(M, A),$$

i.e., $\nu^*(f^* \times g^*) = g^*\mu^*$. Then all the hypotheses of Lemma 4.8, in the following diagram, are satisfied. Therefore, ν^* is a semigroup action. It is clear that $\nu = \nu^* \mid S \times Y$, and hence ν is a multi-action.

$$\begin{tikzcd}
2^T \times 2^X \arrow[r, "\mu^*"] \arrow[d, "f^* \times g^*"'] & 2^X \arrow[d, "g^*"] \\
2^S \times 2^Y \arrow[r, "\nu^*"'] & 2^Y
\end{tikzcd}$$

<u>Theorem 4.10</u>. Suppose T multi-acts on a space X, via μ, with T and X compact. Let $\mathcal{C}$ be a closed multi-congruence on T with $T/\mathcal{C}$ closed in 2^T and let $\mathcal{E}$ be a closed equivalence relation on X with $X/\mathcal{E}$ closed in 2^X such that $M(tx) \cap X/\mathcal{E} = M(sy) \cap X/\mathcal{E}$ whenever $(t, s) \in \mathcal{C}$ and $(x, y) \in \mathcal{E}$. Then $T/\mathcal{C}$ multi-acts on $X/\mathcal{E}$ via $\nu(p(t), q(x)) = M(tx) \cap X/\mathcal{E}$ where p is the quotient map from T onto $T/\mathcal{C}$ and q is the quotient map from X onto $X/\mathcal{E}$.

Proof. Let $\nu: T/\mathcal{C} \times X/\mathcal{E} \longrightarrow 2^{X/\mathcal{E}}$ be defined by $\nu(p(t), q(x)) = M(tx) \cap X/\mathcal{E}$. Then ν is well defined by the assumption. Now let $(t, x) \in T \times X$. Then

$$\nu(p \times q)(t, x) = \nu(p(t), q(x)) = M(tx) \cap X/\mathcal{E} = q^* \mu(t, x),$$

i.e., $\nu(p \times q) = q^*\mu$ so that the following diagram is analytic.

$$\begin{tikzcd} T \times X \arrow[r, "\mu"] \arrow[d, "p \times q"'] & 2^X \arrow[d, "q^*"] \\ T/\mathcal{C} \times X/\mathcal{E} \arrow[r, "\nu"'] & 2^{X/\mathcal{E}} \end{tikzcd}$$

Thus all the hypotheses of Theorem 4.9 are satisfied, and ν is a multi-action.

Definition 4.11. Suppose T multi-acts on a space X. An equivalence relation $\mathcal{E}$ on X is called a T-equivalence relation if and only if $M(tx) \cap X/\mathcal{E} = M(ty) \cap X/\mathcal{E}$ for each $(x, y) \in \mathcal{E}$ and each $t \in T$. (see [23])

Corollary 4.12. Suppose T multi-acts on X, via μ, with T and X compact. Let $\mathcal{E}$ be a closed T-equivalence relation on X with $X/\mathcal{E}$ closed in 2^X. Then T

multi-acts on $X/\mathcal{E}$ via $\nu(t, q(x)) = M(tx) \cap X/\mathcal{E}$.

Corollary 4.13. Suppose T multi-acts on X, via μ, with T and X compact. Let $\mathcal{C}$ be a closed multi-congruence on T such that $tx = sx$ for each $x \in X$ whenever $(t, s) \in \mathcal{C}$. If $T/\mathcal{C}$ is closed in 2^T, then $T/\mathcal{C}$ multi-acts on X via $\nu(p(t), x) = tx$.

Lemma 4.14. Suppose T multi-acts on a space X. Let J be a closed ideal of T, $\mathcal{C} = J \times J \cup \Delta$, and $\mathcal{E} = JX \times JX \cup \Delta$. Then $M(tx) \cap X/\mathcal{E} = M(sy) \cap X/\mathcal{E}$ whenever $(t, s) \in \mathcal{C}$ and $(x, y) \in \mathcal{E}$.

Proof. If $t \in J$, then $s \in J$ so that $tx \subset JX$ and $sy \subset JX$, i.e., $M(tx) \cap X/\mathcal{E} = M(xy) \cap X/\mathcal{E}$. If $t \notin J$, then $t = s$. If $x \notin JX$ then $x = y$ so that $tx = sy$. If $x \in JX$ then $tx \subset TJX \subset JX$ and $sy \subset JX$. Thus, in either case, $M(tx) \cap X/\mathcal{E} = M(xy) \cap X/\mathcal{E}$.

Theorem 4.15. Suppose T multi-acts on a space X, via μ, with T and X compact. Let J be a closed ideal of T with $T/[J]$ and $X/[JX]$ closed in 2^T and 2^X respectively. Then $T/[J]$ multi-acts on X/JX via

$$\nu(p(t), q(x)) = M(tx) \cap X/[JX].$$

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BIOGRAPHICAL SKETCH

Younki Chae was born March 26, 1933, at Taegu, Korea and graduated from Taegun High School in 1952. In 1958, he received the degree of Bachelor of Science with a major in Mathematics from Kyungpock National University. Two years later, he received the degree of Master of Science with a major in Mathematics from Kyungpook National University. From 1960 until 1967, he was a member of the faculty at Kyungpook National University. In 1967 he enrolled in the Graduate School of the University of Florida. He worked as an interim instructor in the Department of Mathematics from May, 1967 until the present time, and during the same time pursued his work toward the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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